Distributed Convex Optimization in Networks of Agents with Single-integrator Dynamics

A. Adibzadeh, M. Zamani, A. A. Suratgar , M. B. Menhaj

Dept. of Electrical Engineering, Amirkabir University of Technology, Tehran, Iran.

School of Electrical Engineering and Computer Science, The University of Newcastle, Callaghan, NSW, Australia.

INTRODUCTION

In recent years, developing distributed paradigms for solving optimization problems among interconnected agents has attracted attention of researchers.

In this paper, distributed convex optimization problem over undirected dynamical networks is studied. Here, networked agents are supposed to rendezvous at a point that is the solution of a global convex optimization problem with some local inequality constraints. To this end, all agents shall cooperate with their neighbors to seek the optimum point of the network's global objective function. A consensus-based distributed optimization algorithm is proposed, which combines an interior-point optimization algorithm with a nonlinear consensus protocol to find the optimum value of the global objective function. We tackle this problem by addressing its subproblems, namely a consensus problem and a convex optimization problem. Firstly, we propose a saturation protocol for the consensus subproblem. Then, to solve the distributed optimization part, we implement an updating rule, which yields the optimum value of the global objective function, with the help of local estimators in a distributed fashion. Convergence analysis for the proposed protocol based on the Lyapunov stability theory for time-varying nonlinear systems is included. A simulation example is given at the end to illustrate the effectiveness of the proposed algorithm.

PROBLEM STATEMENT

consider the following autonomous agents under the topology *G*. Each agent is described by the continuous-time single-integrator dynamics:

$$\dot{x}_i(t) = u_i(t), \quad i \in N$$

Where $u_i(t) \in \mathbb{R}$ and $x_i(t) \in \mathbb{R}$ denote the state and control input to the agent *i*, respectively. Here, we consider only one dimensional agents for the sake of simplicity in notations. However, it is straightforward to show that our algorithm can be extended to higher dimensional dynamics as each dimension is decoupled from others and can be treated independently.

The agents are supposed to rendezvous at a point that shall minimize the aggregate convex function

$$\sum_{i=1}^{N} f_i(x)$$

with regards to individual convex inequalities

$$g_i(x) \le 0, \; i = 1, \ldots, N$$

This problem can be written by

$$\min_x F(x) = \sum_{i=1}^{N} f_i(x),$$

$$\text{subject to } g_i(x) \le 0, i \in \mathrm{N} \,,$$

It is supposed that each agent only has the information of its own local objective function and states of those agents within the set of its neighbors. We express the above explained problem as the following convex optimization problem,

$$\min_{\substack{x_i \\ i=1,\ldots,N}} \sum_{i=1}^{N} f_i(x_i),$$

$$\text{subject to} \begin{cases} g_i(x_i) \le 0, i \in \mathrm{N} \,, \\ x_i = x_j, \forall i, j \in \mathrm{N} \,. \end{cases}$$

In the above minimization problem, the consensus constraint, is imposed to guarantee that the same decision is made by all agents eventually. It is assumed that all the objective functions are strictly convex ,and the slaters' condition hold.

SOLUTION

In this subsection, we propose a central paradigm to find the solution of the problem. Later, in the next subsection, we realize this centralized protocol via a distributed algorithm.

A. Centralized Algorithm

We use law the interior-point method and propose the following centralized control to find the optimal solution for the optimization problem,

$$u_i(t) = -\left(\sum_{i=1}^{N} \frac{\partial^2 L_i(x_i,t)}{\partial x_i^2}\right)^{-1} \left(\sum_{i=1}^{N} \frac{\partial L_i(x_i,t)}{\partial x_i} + \sum_{i=1}^{N} \frac{\partial^2 L_i(x_i,t)}{\partial x_i \partial t}\right) + r_i \quad (*)$$

where

$$L_i(x_i,t) = f_i(x_i) - \frac{\alpha}{t+1} ln\left(-g_i(x_i)\right)$$

and

$$r_i = -\beta_1 \sum_{j \in \mathrm{N}_i} tanh \beta_2 (x_i - x_j), \;\; \beta_1, \beta_2 \in \mathbb{R}^+.$$

It can be shown that the that the positions of agents reach consensus under the above control law. Furthermore, it yields the solution to the optimization problem.

Note that the above control command consists of two parts: the first term is to minimize the local objective function, and the second part is a saturation term associated with the consensus error.

B. Distributed Algorithm

It is obvious that the control law is not locally implementable since it requires the knowledge of the whole network. Through the following algorithm, we estimate (*) and adopt it to solve the optimization problem in a distributed manner.

As it follows, each agent generates an internal dynamics to obtain the estimates of collective objective function's gradients and other terms in a cooperative fashion. Consider the following estimator dynamics,

$$\dot{\kappa}_i(t) \;\; = -c \sum_{j \in \mathrm{N}_i} sgn\left(v_i(t) - v_j(t)\right),$$

where

$$v_i(t) = \kappa_i(t) + \begin{bmatrix} \frac{\partial L_i(x_i,t)}{\partial x_i} \\ \frac{\partial^2 L_i(x_i,t)}{\partial x_i \partial t} \\ \frac{\partial^2 L_i(x_i,t)}{\partial x_i^2} \end{bmatrix}.$$

It can be shown that if under some mild conditions consensus on v_i is achieved over a finite time, say *T*. Thus, the following holds:

$$v_i(t) = \frac{1}{N} \sum_{i=1}^{N} \begin{bmatrix} v_{i1} \\ v_{i2} \\ v_{i3} \end{bmatrix}.$$

where $v_{i1} = \frac{\partial L_i(x_i,t)}{\partial x_i}, v_{i2} = \frac{\partial^2 L_i(x_i,t)}{\partial x_i \partial t}$, and $v_{i3} = \frac{\partial^2 L_i(x_i,t)}{\partial x_i^2}$.

The main result of this paper is given in the following:

If $\sum_{i=1}^{N} \kappa_i(0) = 0$ and $c > \sup_t \left\{ \left\| \kappa_i(x_i,t) \right\|_\infty \right\}, \forall i \in \mathrm{N}$

,then, the protocol

$$u_i(t) = -v_{i3}^{-1}\left(v_{i1} + v_{i2}\right) + r_i, \; i = 1, \ldots, N$$

will drive the agents to the solution of the distributed convex optimization problem.

CONCLUSION

We investigated the problem of distributed optimization for undirected networks of single-integrator agents. A centralized control law, which yields the optimal solution to this problem, and consists of a saturation consensus part and an optimization part based on the interior-point method was proposed. To illustrate the convergence of the proposed algorithm, we first established that the proposed consensus protocol provides practical consensus, i.e. all agents will have the same decision eventually, perhaps with a small admitted error. We then suggested a distributed estimator as a tool to estimate some terms within the protocol associated with the global knowledge, which is only partially available to the agents. It was proved that the presented distributed algorithm converges to the solution of the original constrained convex optimization problem. Finally, to evaluate the performance of our work, a numerical example was presented.

Amirkabir
University of Technology